Exercise 12G: Solutions

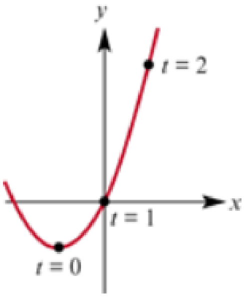
1 a From the first equation we know that $t = x + 1$. Substitute this into the second equation to get

$$\begin{aligned} y &= (x+1)^2 - 1 \\ &= x^2 + 2x. \end{aligned}$$

b To sketch the curve it helps to write $y = x(x+2)$. This is a parabola with intercepts at $x = 0$ and $x = -2$. To label the points corresponding to $t = 0, 1, 2, 3$, we first complete the table shown below.

t	0	1	2
$x = t - 1$	-1	0	1
$y = t^2 - 1$	-1	0	3

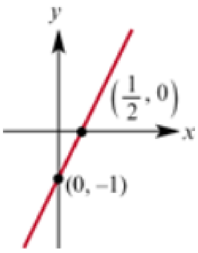
The curve and the required points are shown below.



2 a From the first equation we know that $t = x - 1$. Substitute this into the second equation to get

$$\begin{aligned} y &= 2t + 1 \\ &= 2(x-1) + 1 \\ &= 2x - 2 + 1 \\ &= 2x - 1. \end{aligned}$$

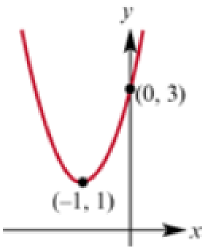
We obtain straight line whose equation is $y = 2x - 1$, and whose graph is shown below.



b From the first equation we know that $t = x + 1$. Substitute this into the second equation to get

$$\begin{aligned} y &= 2t^2 + 1 \\ &= 2(x+1)^2 + 1. \end{aligned}$$

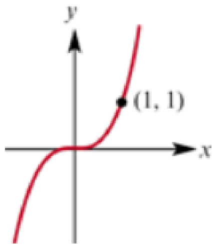
We obtain a parabola whose equation is $y = 2(x+1)^2 + 1$, and whose graph is shown below.



c For this question, we note that $y = (t^2)^3$. Therefore,

$$y = (t^2)^3 = x^3.$$

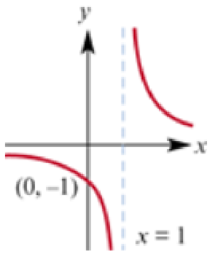
This is clearly a cubic equation whose graph is shown below.



d From the first equation we know that $t = x - 2$. Substitute this into the second equation to get

$$\begin{aligned} y &= \frac{1}{t+1} \\ &= \frac{1}{x-2+1} \\ &= \frac{1}{x-1} \end{aligned}$$

We obtain a hyperbola who=se equation is $y = \dfrac{1}{x-1}$, and whose graph is shown below.



3 a We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This means that

$$\frac{x}{2} = \cos t \text{ and } \frac{y}{2} = \sin t.$$

We then use the Pythagorean identity to show that

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = \cos^2 t + \sin^2 t = 1.$$

Multiplying both sides by 2^2 gives the cartesian equation as

$$x^2 + y^2 = 2^2,$$

which is a circle centred at the origin on radius 2.

b We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This means that

$$\frac{x+1}{3} = \cos t \text{ and } \frac{y-2}{2} = \sin t.$$

We then use the Pythagorean identity to show that

$$\left(\frac{x+1}{3}\right)^2 + \left(\frac{y-2}{2}\right)^2 = \cos^2 t + \sin^2 t = 1.$$

That is,

$$\frac{(x+1)^2}{3^2} + \frac{(y-2)^2}{2^2} = 1,$$

which is an ellipse centred at the point $(-1, 2)$.

c We divide both sides of the equation by 9 so that the equation becomes,

$$\left(\frac{x+3}{3}\right)^2 + \left(\frac{y-2}{2}\right)^2.$$

We then let

$$\cos t = \frac{x+3}{3} \text{ and } \sin t = \frac{y-2}{3}.$$

Therefore, the required equations are

$$x = 3\cos t - 3 \text{ and } y = 3\sin t + 2.$$

d We write this equation as

$$\left(\frac{x+2}{3}\right)^2 + \left(\frac{y-1}{2}\right)^2.$$

We then let

$$\cos t = \frac{x+2}{3} \text{ and } \sin t = \frac{y-1}{2}.$$

so that

$$x = 3\cos t - 2 \text{ and } y = 2\sin t + 1.$$

4 The gradient of the line through points A and B is

$$m = \frac{4-(-2)}{1-(-1)} = \frac{6}{2} = 3.$$

Therefore, the line has equation

$$\begin{aligned} y - 4 &= 3(x-1) \\ y &= 3x + 1. \end{aligned}$$

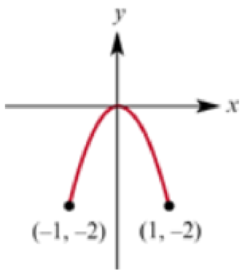
We can simply let $x = t$ so that $y = 3t + 1$. Note that this is not the only possible answer.

5 a From the first equation we know that $t = x + 1$. Substitute this into the second equation to get

$$\begin{aligned} y &= -2t^2 + 4t - 2 \\ &= -2(x+1)^2 + 4(x+1) - 2 \\ &= -2(x^2 + 2x + 1) + 4x + 4 - 2 \\ &= -2x^2 - 4x - 2 + 4x + 4 - 2 \\ &= -2x^2 \end{aligned}$$

Moreover, since $0 \le t \le 2$, we know that $-1 \le x \le 1$.

b We sketch the curve over the domain $-1 \le x \le 1$.



6 The cartesian equation of the circle is

$$x^2 + y^2 = 1. \quad (1)$$

It is a little harder to find the cartesian equation of the straight line. Solving both equations for t gives,

$$t = \frac{x-6}{3} \text{ and } t = \frac{y-8}{4}.$$

Therefore,

$$\begin{aligned} \frac{x-6}{4} &= \frac{y-8}{4} \\ 4(x-6) &= 3(y-8) \\ 4x - 24 &= 3y - 24 \\ y &= \frac{4x}{3} \quad (2) \end{aligned}$$

Solving equations (1) and (2) simultaneously gives $x = -\frac{3}{5}$ and $x = \frac{3}{5}$. Substituting these two values into the equation $y = \frac{4x}{3}$ gives $y = -\frac{4}{5}$ and $x = \frac{4}{5}$ respectively. Therefore, the required coordinates are $\left(-\frac{3}{5}, -\frac{4}{5}\right)$ and $\left(\frac{3}{5}, \frac{4}{5}\right)$.

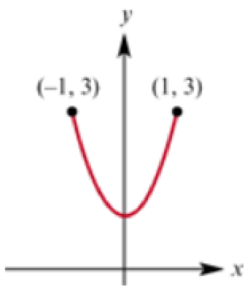
7 a We substitute $x = \sin t$ into the second equation to give,

$$\begin{aligned} y &= 2\sin^2 t + 1 \\ &= 2x^2 + 1. \end{aligned}$$

b Since the domain is the set of possible x-values and $x = \sin t$ where $0 \le t \le 2\pi$, the domain will be $-1 \le x \le 1$.

c Since the domain is the set of x such that $-1 \le x \le -1$, the range must be the set of y such that $1 \le y \le 3$.

d The curve is sketched below over the interval $-1 \le x \le 1$.



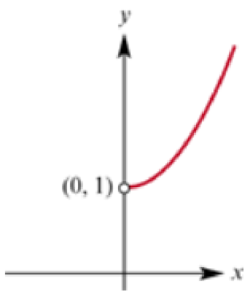
8 a We substitute $x = 2^t$ into the second equation to give,

$$\begin{aligned} y &= 2^{2t} + 1 \\ &= (2^t)^2 + 1 \\ &= x^2 + 1. \end{aligned}$$

b The domain is the set of possible x-values. Since $x = 2^t$ and $t \in \mathbb{R}$, we know that the domain will be $x > 0$.

c Since the domain is the set of all x such that $x > 0$, the range must be the set of y such that $y > 1$.

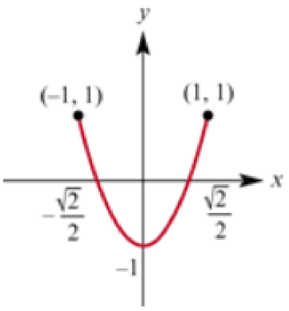
d The curve is sketched below over the interval $x > 0$.



9 Here, we must make use of the identity $\cos^2 t + \sin^2 t = 1$. Since $x = \cos t$ we have,

$$\begin{aligned} y &= 1 - 2\sin^2 t \\ &= 1 - 2(1 - \cos^2 t) \\ &= 1 - 2 + 2\cos^2 t \\ &= -1 + 2x^2 \end{aligned}$$

The domain is the set of possible x-values. Since $x = \cos t$ and $0 \le t \le 2\pi$, we know that the domain will be $-1 \le x \le 1$. We sketch the curve over this interval.

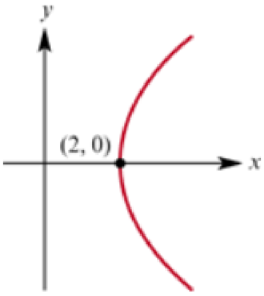


10a We substitute $x = 2^t + 2^{-t}$ and $y = 2^t - 2^{-t}$ into the left hand side of the cartesian equation. This gives,

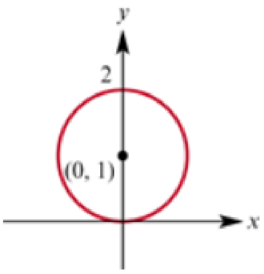
$$\begin{aligned}\text{LHS} &= \frac{x^2}{4} - \frac{y^2}{4} \\ &= \frac{(2^t + 2^{-t})^2}{4} - \frac{(2^t - 2^{-t})^2}{4} \\ &= \frac{2^{2t} + 2 + 2^{-2t}}{4} - \frac{(2^{2t} - 2 + 2^{-2t})}{4} \\ &= \frac{2^{2t} + 2 + 2^{-2t} - 2^{2t} + 2 - 2^{-2t}}{4} \\ &= \frac{4}{4} \\ &= 1 \\ &= \text{RHS},\end{aligned}$$

as required.

b The curves is one side of a hyperbola centred at the origin.



11a This is the equation of a circle of radius 1 centred at $(0, 1)$. Its graph is shown below.



b Since $x = \cos t$ and $y - 1 = \sin t$, we have

$$x^2 + (y-1)^2 = \cos^2 t + \sin^2 t = 1.$$

c We will find the points of intersection of the line,

$$y = 2 - tx \quad (1)$$

and the circle,

$$x^2 + (y-1)^2 = 1. \quad (2)$$

Substituting equation (1) into equation (2), we find that,

$$\begin{aligned}x^2+(2-tx-1)^2&=1\\x^2+(1-tx)^2&=1\\x^2+1-2tx+t^2x^2&=1\\(1+t^2)x^2-2tx&=0\\x((1+t^2)x-2t)&=0\end{aligned}$$

Since $x \neq 0$, we see that

$$x=\frac{2t}{1+t^2}.$$

We can find y by substituting this into equation (1). This gives,

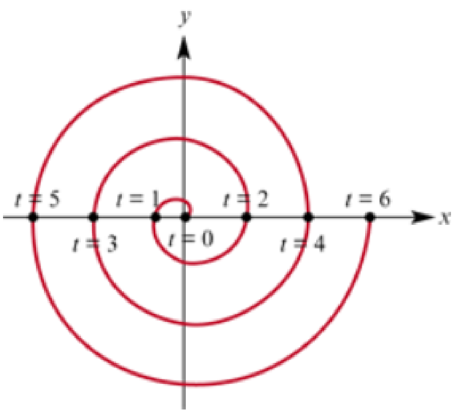
$$\begin{aligned}y&=2-tx\\&=2-\frac{2t^2}{1+t^2}\\&=\frac{2(1+t^2)}{1+t^2}-\frac{2t^2}{1+t^2}\\&=\frac{2}{1+t^2}.\end{aligned}$$

d To verify that these equations parameterise the same circle we note that

$$\begin{aligned}x^2+(y-1)^2&=\left(\frac{2t}{1+t^2}\right)^2+\left(\frac{2}{1+t^2}-1\right)^2\\&=\left(\frac{2t}{1+t^2}\right)^2+\left(\frac{2}{1+t^2}-\frac{1+t^2}{1+t^2}\right)^2\\&=\left(\frac{2t}{1+t^2}\right)^2+\left(\frac{1-t^2}{1+t^2}\right)^2\\&=\frac{4t^2}{(1+t^2)^2}+\frac{(1-t^2)^2}{(1+t^2)^2}\\&=\frac{4t^2}{(1+t^2)^2}+\frac{(1-2t^2+t^4)}{(1+t^2)^2}\\&=\frac{t^4+2t^2+1}{(1+t^2)^2}\\&=\frac{(1+t^2)^2}{(1+t^2)^2}\\&=1,\end{aligned}$$

as required.

12a



b The points corresponding to

$$t=0,1,2,3,4,5,6$$

are all on the x-axis. The values of t correspond to the number of half turns through which the spiral has turned.